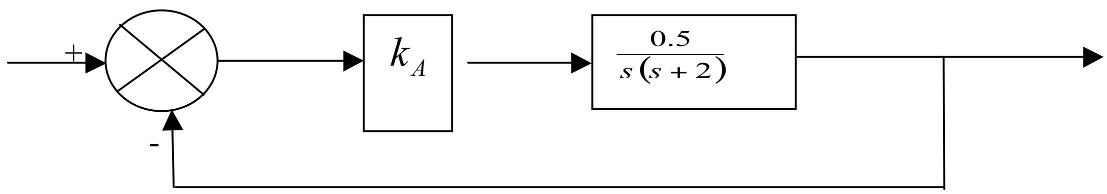
Second Order Systems

A Proportional Controller

In this problem we will consider a controlled second order process. The type of controller that we will use is known as a proportional controller, because the controller's response is proportional to the system error. A block diagram for the system is shown below. Note that the gain of the controller is not yet specified. One of the goals of this problem will be to consider the response of the system to different values of k_A.



In this loop the forward path is determined by multiplying the two blocks together to get

$$\frac{0.5 \bullet k_A}{s(s+2)}$$

where $\qquad k_A \;=\;$ the gain of the proportional controller

The transfer function of the feedback portion of this diagram is "1." The is the equivalent of a very "fast" sensor, in comparison to the rest of the process. Then, the "open loop gain" for this diagram is

$$\frac{0.5 \bullet k_A}{s(s+2)}$$

The overall transfer function *for a negative feedback loop* is then computed as

$$G_{overall} = \frac{\text{Forward Path}}{1 + \text{Open Loop Gain}}$$

If the process is a *positive feedback loop,* the overall transfer function is:

$$G_{overall} = \frac{\text{Forward Path}}{1 - \text{Open Loop Gain}}$$

Since our process is a negative feedback loop

$$G(s) = \frac{\frac{0.5 \bullet k_A}{s(s+2)}}{1 + \frac{0.5 \bullet k_A}{s(s+2)}}$$

or

$$G(s) = \frac{\left(\frac{0.5 \bullet k_A}{s(s+2)}\right)}{\left(\frac{s(s+2) + 0.5 \bullet k_A}{s(s+2)}\right)}$$

Simplifying the equation yields:

$$G(s) = \frac{0.5\ k_A}{s^2 + 2s + 0.5\ k_A}$$

This transfer function is now in the standard format for second order processes, which is

$$G(s) = \frac{\omega_N^{\ 2}}{s^2 + 2\zeta\omega_N^{\ s} + \omega_N^{\ 2}}$$

where: ω_N = the "undamped" or "natural" frequency of the process (cycles/minute)
= $\sqrt{0.5k_A}$
ζ = the damping ratio [=] dimensionless

For the our problem

$$\omega_n = \sqrt{0.5k_A}$$

and

$$2\zeta\omega_n = 2$$

so

$$\zeta = \frac{1}{\omega_N}$$

or

$$\zeta = \frac{1}{\sqrt{0.5 k_A}}$$

If we consider the process' response to a unit step change, we multiply G(s) by R(s) to get

$$C(s) = \frac{1}{s} \bullet \frac{0.5\, k_A}{\left(s^2 + 2s + 0.5\, k_A\right)}$$

The time domain solution to a second order polynomial in the denominator has three possible solutions, depending on the roots of the polynomial. If we consider the general case, solving for "s" using the quadratic equation yields

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where:

$$a = 1$$
$$b = 2\zeta\omega_n$$
$$c = \omega_n^2$$

$$s = \frac{-2\zeta\omega_N \pm \sqrt{4\zeta^2\omega_N^{\ 2} - 4(1)\omega_N^{\ 2}}}{2(1)}$$

$$s = \frac{-2\zeta\omega_N \pm 2\sqrt{\zeta^2\omega_N^{\ 2} - \omega_N^{\ 2}}}{2}$$

$$s = \zeta\omega_N \pm \omega_N\sqrt{\zeta^2 - 1}$$

ω_n must be ≥ 0, since you can not have a negative frequency. In addition, $\omega_n \neq 0$. Therefore, the behavior of the system is dependant on the value of ζ.

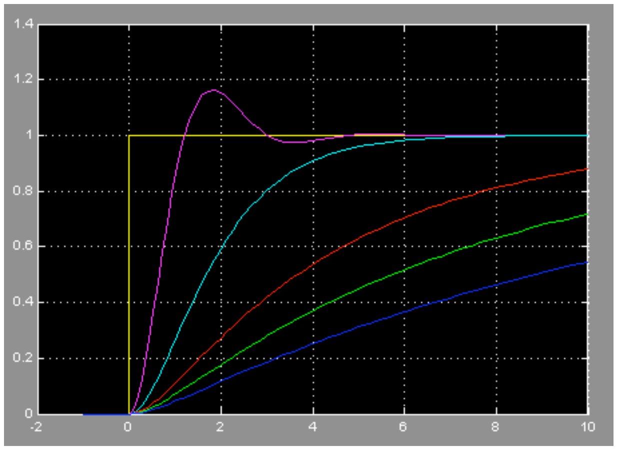
$\zeta = 0 \Rightarrow$ undamped sinusoid (purely imaginary roots)

$0 < \zeta < 1 \Rightarrow$ damped sinusoid resulting (complex roots)

$\zeta = 1 \Rightarrow$ critically damped (real and equal roots)

$\zeta^2 > 1 \Rightarrow$ overdamped response (real roots)

Since, in our problem, ζ is variable and a function of k_A, we can control the response of the system by varying k_A. The plot below shows the system's responses for a variety of values of ζ (0.5, 1, 1.5, 2, 2.5 from top to bottom).



Critically Damped Solution

For example, if we want the system to respond to the step change input as quickly as possible without any overshoot, then we would like the system to be critically damped. This is the light blue curve in the plot above For this case,

$$\zeta = 1$$

and (by the way)

$$\omega_N = \frac{1}{\zeta} = 1$$

so

$$\zeta = 1 = \frac{1}{\sqrt{0.5k_A}}$$

and

$$0.5k_A = 1^2$$

$$k_A = 2$$

In practice, you would seldom design for critically damped, since minor instrumentation errors are likely to result in the overshoot. If overshoot must be avoided, you would probably design for a slightly overdamped system.

How long would it take for our process to reach steady-state, based on our k_A. For a 2^{nd} order system,

$$\tau = \frac{1}{\zeta\omega_N}$$

so

$$\tau = \frac{1}{1 \bullet 1} = 1$$

As with simple first order processes, this process never comes to steady-state, so we must define "steady-state." Typically steady-state is defined as either 4 • τ (98% of the value at t = ∞) or 5 • τ (99% of the value at t = ∞). In this problem, we will use 4 τ 's as our definition of $t_{steady\text{-}state}$. Then, the time to steady state is 4 minutes.

Overshoot Solutions

The above case describes the fastest $t_{steady\text{-}state}$ without overshoot. However, if overshoot is acceptable, we can reach $t_{steady\text{-}state}$ more rapidly. For example, compare response for ζ = 0.5 (the purple line) to the response for ζ = 0.5 (the light blue line) in the plot of varying values of ζ. In this type of problem

$$\text{Maximum Percentage Overshoot} = \frac{C(t_{\text{peak}}) - C(t_{\text{steady-state}})}{C(t_{\text{steady-state}})} \bullet 100\%$$

This overshoot is measured at the first (highest) peak in the process' response. The location of this peak is determined based on the *damped natural frequency,* ω_d, of the system. This frequency differs from the *undamped natural frequency,* ω_n, of the process based on the equation

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

The maximum value occurs at

$$t_{\text{peak}} = \frac{\pi}{\omega_d}$$

at this time

$$\text{Maximum Percentage Overshoot} = e^{-\left(\xi / \sqrt{1-\xi^2}\right)\bullet\pi} \bullet 100\%$$